SELF-SIMILAR MOTION OF A LIQUID UNDER THE ACTION OF SURFACE TENSION

(AVTOMODEL'NOE DVIZHENIE ZHIDKOSTI POD DEISTVIEM POVERKHNOSTNOGO NATIAZHENIIA)

PMM Vol.29, № 1, 1965, pp. 54-61

F.L. CHERNOUS'KO
(Moscow)

(Received October 22, 1964)

The motion of a weightless liquid with a free surface is investigated developing from rest under the action of surface tension forces. Formulation of the self-similar problem and its solution for one case are given.

1. Plane motion of an ideal incompressible weightless liquid with density ρ is examined. Let x and y be orthogonal Cartesian coordinates in the plane of flow. At the instant of time $t = 0$ the liquid is at rest and occupies a wedge (Fig.1) with an angle α. The wedge is bound by the free surface $y = 0$ and by the solid wall $y = -x\tan\alpha$. The coefficient of surface tension σ on the free boundary and the contact angle γ at the boundary of the liquid with the wall (Fig.1) are considered constant. If $\gamma \neq \alpha$, then for time $t > 0$ the liquid will start in motion which, apparently, will be potential. Motion of this type can develop on sudden "turning on" of surface tension and also, for example, in the following case. Let at $t < 0$ the liquid be at rest in the gravitational field. In this case the free surface is substantially different from the plane $y = 0$ only in the region near the wall, where a meniscous is formed [1]. The dimensions of the miniscous will be smaller, the greater the ratio of gravitational force to surface tension forces. Let at the moment $t = 0$ the gravitational force instantaneously become zero. Then for $t > 0$ motion develops which will be close to the self-similar motion examined here if the dimensions of the initial meniscous are small compared to the scale of process which interests us (i.e. if the gravitational force at $t < 0$ was sufficiently large).

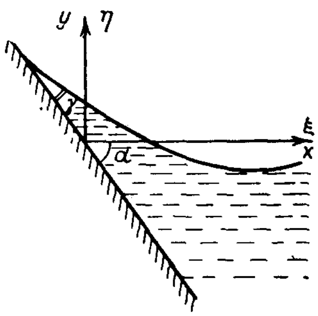

Fig. 1

The potential of velocities $\varphi^\circ(x,y,t)$ satisfies the Laplace equation in the region of the flow and the condition of no flow through the wall (subscripts indicate partial derivatives).

$$\varphi_{xx}{}^\circ + \varphi_{yy}{}^\circ = 0, \qquad \varphi_y{}^\circ + \varphi_x{}^\circ \tan\alpha = 0 \qquad \text{for } y = -x\tan\alpha \tag{1.1}$$

The pressure p in the liquid at the free surface $y = f^\circ(x,t)$ is related [1] to the constant pressure outside the liquid through the relationship

$$p = p_0 - \sigma K, \qquad K = \pm f_{xx}{}^\circ (1 + f_x{}^{\circ 2})^{-3/2} \tag{1.2}$$

Here K is the curvature of surface. The upper sign in Equation (1.2) and also in (1.7) must be used when the liquid is below the free surface (as in Fig.1) and the lower sign in the opposite case.

Keeping in mind the equation for p we write for points on the free surface the Cauchy-Lagrange integral and also the kinematic condition (1.3)

$$\varphi_t{}^\circ + {}^1/_2 (\nabla\varphi^\circ)^2 - \sigma K / \rho = 0, \qquad f_t{}^\circ + f_x{}^\circ \varphi_x{}^\circ - \varphi_y{}^\circ = 0 \qquad \text{for } y = f^\circ(x,t)$$

At the point of contact of the free boundary with the wall we have

$$f_x{}^\circ(x,t) = \tan(\gamma - \alpha) \qquad \text{for } y = f^\circ(x,t) = -x\tan\alpha \tag{1.4}$$

Initial conditions and conditions at infinity have the form

$$\varphi^\circ(x,y,0) = f^\circ(x,0) \equiv 0, \quad \varphi^\circ, f^\circ \to 0 \qquad \text{for } x, y \to \infty \tag{1.5}$$

The problem (1.1) to (1.5) of determining the functions φ° and f° will be self-similar: it contains two dimensional parameters σ and ρ with dimensions $[\sigma] = MT^{-2}$ and $[\rho] = ML^{-3}$. Let us introduce nondimensional independent variables ξ and η and nondimensional unknown functions φ and f.

$$x = \left(\frac{\sigma t^2}{\rho}\right)^{1/3}\xi, \quad y = \left(\frac{\sigma t^2}{\rho}\right)^{1/3}\eta, \quad \varphi^\circ = \left(\frac{\sigma^2 t}{\rho^2}\right)^{1/3}\varphi(\xi,\eta), \quad f^\circ = \left(\frac{\sigma t^2}{\rho}\right)^{1/3} f(\xi) \tag{1.6}$$

Passing to new variables in Equations (1.1) to (1.5) according to (1.6) we obtain the boundary value problem for functions φ and f.

$$\begin{gathered}
\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0, \qquad \varphi_\eta + \varphi_\xi \tan\alpha = 0 \qquad \text{for } \eta = -\xi\tan\alpha \\
{}^1/_3\,\varphi - {}^2/_3(\xi\varphi_\xi + \eta\varphi_\eta) + {}^1/_2(\nabla\varphi)^2 \mp f''(1 + f'^2)^{-3/2} = 0 \\
{}^2/_3(f - \xi f') + f'\varphi_\xi - \varphi_\eta = 0 \qquad \text{for } \eta = f(\xi) \\
f' = \tan(\gamma - \alpha) \qquad \text{for } f(\xi) = -\xi\tan\alpha \\
\varphi(\xi,\eta) \to 0, \qquad f(\xi) \to 0 \qquad \text{for } \xi, \eta \to \infty
\end{gathered} \tag{1.7}$$

Here the prime indicates a derivative with respect to ξ. Nonlinear boundary value problem (1.7) is formulated for the region (Fig.1) bounded by the staight line $\eta = -\xi\tan\alpha$ and the unknown curve $\eta = f(\xi)$. Self-similar axisymmetric motion can be examined in an analogous manner.

2. Problem (1.7) can be linearized if angles γ and α are close to each other, i.e. $\gamma - \alpha = \varepsilon$, $|\varepsilon| \ll 1$. Upon linearization functions φ and f and their derivatives are considered to be small of the order of ε, the condition of the unknown boundary $\eta = f(\xi)$ is reduced to the straight line $\eta = 0$.

From two signs in (1.7) it is necessary to select the upper because for $|\varepsilon| \ll 1$ perturbations of the free boundary are small and the liquid is located everywhere below the free surface.

For the sake of definiteness we take $\alpha = \frac{1}{2}\pi$. Then the linearized boundary value problem is reduced to the determination of the function $\varphi(\xi, \eta)$, harmonic in the quadrant $\xi > 0$, $\eta < 0$, and the function $f(\xi)$ from conditions

$$^1/_3\varphi - {}^2/_3\xi\varphi_\xi - f'' = 0, \quad {}^2/_3(f - \xi f') - \varphi_\eta = 0 \quad \text{for } \eta = 0 \tag{2.1}$$

$$\varphi_\xi = 0 \quad \text{for } \xi = 0, \qquad f'(0) = \varepsilon$$

$$\varphi(\xi, \eta) \to 0, \quad f(\xi) \to 0 \quad \text{for } \xi, \eta \to \infty$$

Differentiating the second condition (2.1) with respect to ξ , we eliminate f'' from conditions for $\eta = 0$ and obtain boundary conditions for φ in the form

$$\varphi_{\xi\eta} - {}^4/_9\xi^2\varphi_\xi + {}^2/_9\xi\varphi = 0 \quad \text{for } \eta = 0$$

$$\varphi_\xi = 0 \quad \text{for } \xi = 0, \qquad \varphi \to 0 \quad \text{for } \xi, \eta \to \infty \tag{2.2}$$

The homogeneous linear boundary value problem (2.2) contains the second derivative $\varphi_{\xi\eta}$ in the boundary condition and is not among the studied types of boundary value problems. As will be shown, this problem has a single-parameter family of solutions and the unique solution is distinguished by the condition $f'(0) = \varepsilon$.

The complex variable $z = \xi + i\eta$ and the complex potential $w = \varphi + i\psi$ are introduced, where ψ is a harmonic function, conjugate with φ . Without loosing generality it is assumed on the basis of (2.2) that $\psi = 0$ for $\xi = 0$. Consequently, the analytic function $w(z)$ can be extended by symmetry to the entire lower half-plane $\eta < 0$. Along the real axis, as follows from (2.2), we obtain (prime indicates derivative with respect to z)

$$\operatorname{Re}(iw'' - {}^4/_9 z^2 w' + {}^2/_9 zw) = 0 \quad \text{for } \eta = 0 \tag{2.3}$$

The natural assumption is made that w tends to 0 as a dipole potential for $z \to \infty$, i.e. $w = O(z^{-1}), w' = O(z^{-2}), w'' = O(z^{-3})$ for $z \to \infty$. This assumption is justified by the fact that a unique solution will be constructed below, which has such asymptotic behavior for $z \to \infty$.

The function under the Re sign in Equation (2.3) is analytical for $\eta < 0$ and, by virtue of the assumption made, it is bounded at infinity. Then it follows from (2.3)

$$iw'' - {}^4/_9 z^2 w' + {}^2/_9 zw = iC \tag{2.4}$$

Here C is so far an arbitrary real constant.

Thus the boundary problem (2.1) is reduced to finding solutions of an ordinary linear differential equation (2.4) for given asymptotic behavior $w = O(z^{-1})$ at infinity.

When the function $w(z) = \varphi + i\psi$ is found, the shape of the free surface $f(\xi)$ is determined from the second condition (2.1) which represents a linear equation of the first order for f. Taking into account the equality $\varphi_\eta = -\psi_\xi$, we write the solution of this equation satidfying the condition $f(\infty) = 0$

$$f(\xi) = -\frac{3\xi}{2}\int_\xi^\infty \frac{\psi_\xi(x, 0)}{x^2}\,dx \qquad (2.5)$$

Differentiating Equation (2.5) and equating $f'(0) = \varepsilon$, we obtain

$$f'(\xi) = -\frac{3}{2}\int_\xi^\infty \frac{\psi_\xi(x, 0)}{x^2}\,dx + \frac{3\psi_\xi(\xi, 0)}{2\xi} = \frac{3}{2}\int_\xi^\infty \frac{\psi_\xi(\xi, 0) - \psi_\xi(x, 0)}{x^2}\,dx$$

$$\frac{3}{2}\int^\infty \frac{\psi_\xi(0, 0) - \psi_\xi(x, 0)}{x^2}\,dx = \varepsilon \qquad (2.6)$$

Integral (2.6) converges for $x = \infty$ by virtue of asymptotic behavior $w = O(z^{-1})$, $\psi_\xi = O(\xi^{-2})$. From symmetry condition $\psi(0, \eta) = 0$ it follows that $\psi(\xi, 0)$ is an odd and $\psi_\xi(\xi, 0)$ is an even function of ξ Therefore $\psi_\xi(0, 0) - \psi_\xi(x, 0) = O(x^2)$ for $x \to 0$ and integral (2.6) converges also for $x = 0$.. Condition (2.6) serves for determination of constant C in (2.4) which will enter as a factor into w. It is easy to verify that for $f(\xi)$ in (2.5) the condition is satisfied which expresses mass conservation of liquid

$$\int_0^\infty f(\xi)\,d\xi = 0$$

3. The particular solution w_0 of the inhomogeneous equation (2.4) and the linearly independent particular solutions w_1 and w_2 of the homogeneous equation corresponding to (2.4) are being sought in the form

$$w_0 = C\sum_{k=0}^\infty a_k z^{3k+2}, \qquad w_1 = \sum_{k=0}^\infty a_k' z^{3k}, \qquad w_2 = \sum_{k=0}^\infty a_k'' z^{3k+1}$$

Substituting each of these series into equations and equating coefficients of powers of z, we obtain recurrent relationships (coefficients a_0' and a_0'' are arbitrary)

$$a_0 = \frac{1}{2}, \qquad \frac{a_k}{a_{k-1}} = \frac{2(-i)(2k-1)}{3(3k+1)(3k+2)}, \qquad \frac{a_k'}{a_{k-1}'} = \frac{2(-i)(2k-{}^7/_3)}{3(3k-1)3k}$$

$$\frac{a_k''}{a_{k-1}''} = \frac{2(-i)(2k-{}^5/_3)}{3\cdot 3k(3k+1)} \qquad (k = 1, 2, \ldots)$$

These equations, as can be readily verified, can be satisfied by taking

$$a_k = \frac{(-i)^k(2k)!}{(3k+2)!}, \qquad a_k' = \frac{(-i)^k\Gamma(2k-{}^1/_3)}{(3k)!}, \qquad a_k'' = \frac{(-i)^{k+1}\Gamma(2k+{}^1/_3)}{(3k+1)!}$$

$$(k = 0, 1, 2, \ldots)$$

Consequently, the desired functions w_0, w_1, w_2 and the general solution w of Equation (2.4) are

$$w_0 = C \sum_{k=0}^{\infty} \frac{(-i)^k (2k)!}{(3k+2)!} z^{3k+2}, \qquad w_1 = \sum_{k=0}^{\infty} \frac{(-i)^k \Gamma(2k - 1/3)}{(3k)!} z^{3k}$$

$$w_2 = \sum_{k=0}^{\infty} \frac{(-i)^{k+1} \Gamma(2k + 1/3)}{(3k+1)!} z^{3k+1}, \qquad w = w_0 + C_1 w_1 + C_2 w_2 \tag{3.1}$$

Series in (3.1) converge for all $z \neq \infty$, i.e. w is an integral function.

Since the desired particular solution satisfies the condition $\operatorname{Im} w = 0$ for $z = i\eta$, the arbitrary constants C_1, C_2 and also C must be real.

For their determination Equation (2.4) is reduced at first by substitution of variables

$$\tau = -{}^4/_{27}\, iz^3, \qquad \arg \tau = 3 \arg z + {}^3/_2\, \pi \tag{3.2}$$

to a degenerate hypergeometric equation

$$\tau \frac{d^2 w}{d\tau^2} + \left(\frac{2}{3} - \tau\right) \frac{dw}{d\tau} + \frac{w}{6} = -\frac{C}{2^{4/3} \tau^{1/3}} \tag{3.3}$$

As linearly independent particular solutions of the homogeneous equation corresponding to (3.3) we take confluent hypergeometric functions [2 and 3]

$$\Phi = \Phi(-{}^1/_6, {}^2/_3; \tau), \qquad \Psi = \Psi(-{}^1/_6, {}^2/_3; \tau) \tag{3.4}$$

The Wronskian of solutions (3.4) is [2]

$$W = \Phi \Psi_\tau' - \Psi \Phi_\tau' = -[\Gamma({}^2/_3) / \Gamma(-{}^1/_6)]\, e^\tau \tau^{-2/3} \tag{3.5}$$

Solutions of the inhomogeneous equations (3.3) are sought by the method of variation of arbitrary constants assuming that

$$w = u\Phi + v\Psi, \qquad w_\tau' = u\Phi_\tau' + v\Psi_\tau' \tag{3.6}$$

As is common in the method of variation of constants the following equations are obtained for functions u and v :

$$\frac{du}{d\tau} = \frac{C\Psi}{2^{4/3} \tau^{1/3} W} = -De^{-\tau} \tau^{-1/3} \Psi$$

$$\frac{dv}{d\tau} = De^{-\tau} \tau^{-1/3} \Phi, \qquad D = \frac{C\Gamma(-{}^1/_6)}{2^{4/3} \Gamma({}^2/_3)} \tag{3.7}$$

In the derivation of relationships (3.7), Equation (3.5) was utilized.

By virtue of (3.2) we have in the region of flow

$$\xi > 0, \ \eta < 0; \qquad -{}^1/_2\, \pi < \arg z < 0; \qquad 0 < \arg \tau < {}^3/_2\, \pi$$

The asymptotic behavior of solution w for $\tau \to \infty$ in the sector $0 < \arg \tau < \frac{1}{2}\pi$ will be found. In the sector mentioned the following asymptotic equations apply [2]

$$\Phi \sim [\Gamma({}^2/_3) / \Gamma(-{}^1/_6)]\, e^\tau \tau^{-5/6}, \qquad \Psi \sim \tau^{1/6} \tag{3.8}$$

We substitute (3.8) into (3.7) and find u and v satisfying asymptotic integrations. Then u and v are substituted into Expression (3.6) for w. We obtain

$$u \sim u(\infty) + De^{-\tau} \tau^{-1/6}, \qquad v \sim v(\infty) - \frac{2D\Gamma({}^2/_3)}{\Gamma(-{}^1/_6)} \tau^{-1/2}$$

$$w \sim -2D \frac{\Gamma({}^2/_3)}{\Gamma(-{}^1/_6)} \tau^{-1/3} + u(\infty)\Phi + v(\infty)\Psi + O(\tau^{-4/3}) \tag{3.9}$$

For w to have the prescribed asymptotic behavior $w = O(z^{-1}) = O(\tau^{-1/3})$ for $\tau \to \infty$, it is necessary to set the constants $u(\infty) = v(\infty) = 0$. Then, taking also into consideration the value of D in (3.7) and the relationship (3.2), we find from (3.9) the asymptotic behavior

$$w \sim (3iC)/(2z) \qquad \text{for } z \to \infty \tag{3.10}$$

Asymptotic behavior (3.10) is also applicable in the remaining part of the flow region which is examined in an analogous fashion.

For functions u and v we obtain from (3.7)

$$u(\tau) = -D\int_{\infty}^{\tau} e^{-\tau}\tau^{-2/3}\Psi\, d\tau, \qquad v(\tau) = D\int_{\infty}^{\tau} e^{-\tau}\tau^{-2/3}\Phi\, d\tau \tag{3.11}$$

The integration paths in (3.11) start at $\tau \to \infty$, $|\arg \tau| < \frac{1}{2}\pi$

The desired solution w is unambiguously determined by Equations (3.6), (3.4) and (3.11). Substituting into Equations (3.11) and (3.6) known [2] expansions of confluent hypergeometric functions (3.4) in series of powers of τ, we find the expansion of function w for small τ

$$w = \left[u(0) + v(0)\frac{\Gamma(^1/_3)}{\Gamma(^1/_6)}\right] + v(0)\frac{\Gamma(-^1/_3)}{\Gamma(-^1/_6)}\tau^{1/3} + \frac{3D\Gamma(-^1/_3)}{2\Gamma(-^1/_6)}\tau^{2/3} + O(\tau)$$

Constants $u(0)$ and $v(0)$ represent by virtue of (3.11) definite integrals along the real half axis (from 0 to ∞). These integrals are computed from equations given on pages 269 to 270 of [2] or on page 874 of [3]. After simple transformations utilizing functional relationships of the Γ-function and the relation (3.2) we finally obtain

$$w = {^1/_2}\, C\Gamma(-^1/_3) + {^1/_2}\, iC\Gamma(^1/_3)\, z + {^1/_2}\, Cz^2 + O(z^3)$$

Comparing this expansion with Equations (3.1) we find the constants $C_1 = +\frac{1}{2}C$ and $C_2 = -\frac{1}{2}C$.

By virtue of (3.1) the desired solution of Equation (2.4) is

$$w(z) = \frac{C}{2}\sum_{k=0}^{\infty}\left\{(-i)^k z^{3k}\left[\frac{\Gamma(2k-{^1/_3})}{(3k)!} + \frac{i\Gamma(2k+{^1/_3})}{(3k+1)!}z + \frac{2(2k)!}{(3k+2)!}z^2\right]\right\} \tag{3.12}$$

Asymptotic behavior of solution (3.12) for $z \to \infty$ is determined by Equation (3.10). Utilizing asymptotic series for confluent hypergeometric functions [2] it is not difficult to obtain also an asymptotic series for w. The final result is given and, just as Equations (3.1), is verified by direct substitution into Equation (2.4)

$$w(z) \sim \frac{3iC}{2}\sum_{k=0}^{\infty}\frac{(-i)^k(3k)!}{(2k+1)!\, z^{3k+1}} \qquad \text{for } z \to \infty \tag{3.13}$$

Thus, the desired solution of Equation (2.4) with the required asymptotic behavior is determined in the form of a series (3.12), converging for all finite z, in the form of asymptotic series (3.13) and also Equations (3.6) and (3.11) through confluent hypergeometric functions (3.4).

4. For final determination of flow and of the form of the free surface it is also necessary to find the constant C. First of all from Equation (3.12) we have

$$w(0) = \varphi(0, 0) = {^1/_2}\, C\Gamma(-^1/_3), \qquad w'(0) = i\psi_\xi(0, 0) = {^1/_2}\, iC\Gamma(^1/_3) \tag{4.1}$$

The following auxiliary function is introduced:

$$P(\xi) = \frac{3}{2C}\int_0^{\xi} \frac{\psi_\xi(0, 0) - \psi_\xi(x, 0)}{x^2}\, dx \tag{4.2}$$

Using Equation (4.1), (4.2) is rewritten in the form

$$P(\xi) = P(\infty) - \frac{3\Gamma(1/3)}{4\xi} + \frac{3}{2C}\int_{\xi}^{\infty} \frac{\psi_\xi(x, 0)}{x^2}\, dx \tag{4.3}$$

For the function $\psi_\xi(\xi, 0) = \operatorname{Im} w'(\xi)$ it is easy to obtain from Equations (3.12) and (3.13) a converging as well as asymptotic series. Substituting the first of these series into (4.2) and the second into (4.3) we find for P converging and asymptotic (for $\xi \to \infty$) series

$$P(\xi) = \frac{3}{4}\sum_{k=0}^{\infty}\left\{(-1)^k \xi^{6k+1}\left[\frac{\Gamma(4k + 5/3)}{(6k+1)(6k+2)!} + \right.\right.$$
$$\left.\left. + \frac{2(4k+2)!\,\xi^2}{(6k+3)(6k+4)!} + \frac{\Gamma(4k + 13/3)\,\xi^4}{(6k+5)(6k+6)!}\right]\right\}$$

$$P(\xi) \sim P(\infty) - \frac{3\Gamma(1/3)}{4\xi} - \frac{3}{4}\sum_{k=0}^{\infty} \frac{(-1)^k (6k+1)!}{(2k+1)(4k+1)!\,\xi^{6k+3}} =$$
$$= P(\infty) - \frac{3\Gamma(1/3)}{4\xi} - \frac{3}{4\xi^3} + O(\xi^{-9}) \qquad (\xi \to \infty) \tag{4.4}$$

Comparing Equations (2.5), (2.6) and (4.2), (4.3) the constant C and the form of the free surface $f(\xi)$ are represented by the function $P(\xi)$

$$C = \frac{\varepsilon}{P(\infty)}, \qquad f(\xi) = \varepsilon\left\{-\frac{3\Gamma(1/3)}{4P(\infty)} + \xi\left[1 - \frac{P(\xi)}{P(\infty)}\right]\right\} \tag{4.5}$$

Determination of constant C which enters as a multiplier in Equations (3.12) and (3.13) for w , and also the determination of the function $f(\xi)$ are reduced (as can be seen from (4.5)) to calculation of function $P(\xi)$ and in particular $P(\infty)$. For this, series (4.4) may be utilized.

Another method used in this work consists in the following. Let us examine Equation (2.4) along the real axis assuming that in it

$$z = \xi, \; w = \varphi + i\psi = C(y_1 + iy_2).$$

Separating in (2.4) the real and imaginary parts a system of two second order equations is obtained for functions $y_1(\xi)$ and $y_2(\xi)$

$$y_1'' = (4/9)\,\xi^2 y_2' - (2/9)\,\xi y_2 + 1, \qquad y_2'' = (2/9)\,\xi y_1 - (4/9)\,\xi^2 y_1'$$

This system was integrated numerically on an electronic computer from $\xi = 0$ to $\xi = 20$ for initial conditions which were obtained from (3.12)

$$y_1 = 1/2\,\Gamma(-1/3), \qquad y_1' = y_2 = 0, \;\; y_2' = 1/2\,\Gamma(1/3) \qquad \text{for } \xi = 0$$

Function $P(\xi)$ is determined through y_2 by the quadrature (4.2)

$$P(\xi) = \frac{3}{2}\int_0^{\xi} \frac{y_2'(0) - y_2'(x)}{x^2}\, dx = \frac{3}{4}\int_0^{\xi} \frac{\Gamma(1/3) - 2y_2'(x)}{x^2}\, dx \tag{4.6}$$

Indeterminateness of the expression under the integral for $x = 0$ is readily discovered and for small ξ we have from (4.4)

$$P(\xi) = {}^3/_8\,\Gamma\,({}^5/_3)\,\xi + O\,(\xi^3)$$

A check of the determination of $P(\xi)$ was made by means of computing the converging sries (4.4).

The value of $P(\infty)$ was computed through $P(\xi)$ for $\xi = 15$ to 20 by means of the last of Equations (4.4). The function $f(\xi)$ was determined through $P(\xi)$ according to Equation (4.5).

Some results of calculations are presented in which Equations (2.1), (4.1) and (4.5) were also utilized

$$P(\infty) = 2.356, \qquad C = \varepsilon \,/\, P(\infty) = 0.4244\varepsilon$$

$$f(0) = -{}^3/_4\,\varepsilon\Gamma\,({}^1/_3)\,/\,P(\infty) = -0.8527\varepsilon, \quad f'(0) = \varepsilon$$

$$f''(0) = {}^1/_3\,\varphi\,(0, 0) = {}^1/_6\,\varepsilon\Gamma\,(-{}^1/_3)\,/\,P(\infty) = -0.2874\varepsilon$$

$$V = |\,w'(0)\,| = {}^1/_2\,\varepsilon\Gamma\,({}^1/_3)\,/\,P(\infty) = -{}^2/_3 f(0) = 0.5685\,\varepsilon$$

Here V is the module of nondimensional liquid velocity at the origin of coordinates (the velocity here is directed along the η-axis). A plot of the function $\varepsilon^{-1}f(\xi)$ (of the free surface of the liquid) is shown in Fig.2. Oscillations with increasing frequency and rapidly decreasing amplitude along increasing ξ are evident in the graph. These oscillations correspond to capillary waves spreading over the free surface of the liquid according to the self-similarity principle. Shorter waves spread with greater velocity in agreement with general properties of capillary waves [1]. For $\xi \gg 1$ the function $\varepsilon^{-1}f(\xi)$ reaches an asymptote which follows from (4.4) and (4.5)

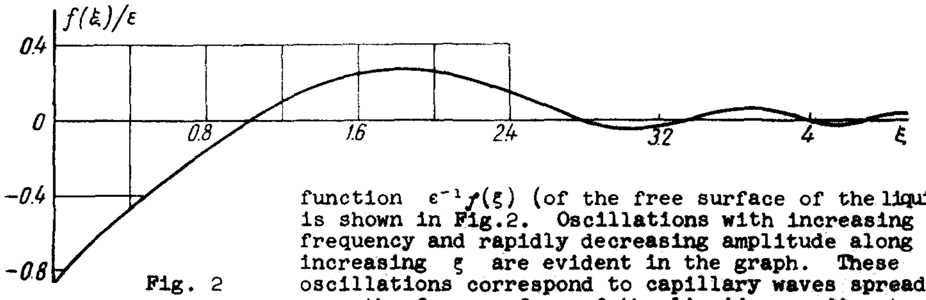

Fig. 2

$$\varepsilon^{-1} f(\xi) \sim {}^3/_4\,[P(\infty)\,\xi^2]^{-1}$$

and it tends to zero while remaining positive.

Thus, the solution of the linearized self-similar equation at $\alpha = \frac{1}{2}\pi$ is completely defined. It is recalled that $\varepsilon = \gamma - \frac{1}{2}\pi$ and therefore $\varepsilon > 0$ for nonwetting liquid and $\varepsilon < 0$ for wetting liquid. Transition to dimensional variables is given by Equations (1.6), for example, the rise of liquid and its velocity near the wall are

$$f^\circ(0, t) = \left(\frac{\sigma t^2}{\rho}\right)^{1/3} f(0) = -\,0.8527\left(\gamma - \frac{\pi}{2}\right)\left(\frac{\sigma t^2}{\rho}\right)^{1/3}$$

$$v^\circ = \frac{\partial f^\circ(0, t)}{\partial t} = \frac{2}{3}\left(\frac{\sigma}{\rho t}\right)^{1/3} f(0) = -0.5685\left(\gamma - \frac{\pi}{2}\right)\left(\frac{\sigma}{\rho t}\right)^{1/3}$$

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Translated by J.R.W.